ISCE Training Note No. 1.2

Calculations

J. M. Woodgate F Inst SCE

Why calculate?

Science and engineering are all about measuring and calculating. Without calculations, only artistic talents can be applied to problems. The result may look pretty, but it is unlikely to work!

It's not possible to understand a product specification without understanding what the individual numbers and units mean. And it's often necessary to do calculations with those numbers in order to find out something important, such as a cable size.

Numbers

Few people have problems with adding and subtracting. It's not too difficult to turn round:

$$44 + 37 = 81$$

into:

$$81 - 44 = 37$$

These are equations; rather simple ones but still equations. But when it comes to multiplying and dividing, the picture is often not so rosy. This matters a lot, because very many important calculations are about multiplying and dividing.

Consider: $4 \times 8 = 32$

If we divide BOTH sides of the equation by 2, it is still true:

$$2 \times 8 = 16$$

Of course, we mustn't divide both the 4 AND the 8 by 2, because that would be equivalent to dividing the left side of the equation by 4, not 2. And this works for ANY number, not just 2.

Equally, we could multiply both sides by any number, such as 25:

$$100 \times 200 = 20\,000$$

Now look at: $13 \times 11 = 143$

Divide both sides by 11:

$$13 \times \frac{11}{11} = \frac{143}{11}$$

$$\text{or } 13 \times 11/11 = 143/11$$

But 11/11 = 1, so we get:

$$13 = 143/11$$

We have changed the multiplication sum into a division sum.

If we multiply a number by itself, the result is called the square of that number:

$$11 \times 11 = 121 = 11^2$$

The little 2 ('superscript') is an 'index' or 'exponent'. We can carry on like that:

$$11 \times 11 \times 11 = 1321 = 11^3$$, '11-cubed'.

We can multiply by adding indices: $11^2 \times 11^3 = 11^5$

and we can divide by subtracting them:

$$2^{10}/2^4 = 2^6$$

Consider:

$$3^7/3^7 = 3^0$$

but the answer is obviously 1 as well.

So $3^0 = 1$.

Similarly,

$$3^{½} \times 3^{½} = 3^1 = 3.$$

So $3^{1/2} = \surd 3$ (the square root of 3)

What about 3^{-1}? Well,

$$3^{-1} \times 3^1 = 3^0 = 1,$$

so $3^{-1} = 1/3$

Letters instead of numbers

Some equations are true only for specific numbers;

$$23 \times 65 = 1495$$

But some equations are true for **all** numbers, and we have already seen one:

$$3^0 = 1$$

If you look back at how we found that, you can see that the number doesn't have to be 3; it could be 4789576, or anything. Some other equations are not true for all numbers, but are true for several. We can write these equations with letters instead of numbers to show that they are not specific to a single number or set of numbers. This is – wait for it! – the dreaded **algebra.**

Traditionally, mostly letters x, y, a, b, c, m and n are used for 'a number'. So, for example, we can write:

$$x^0 = 1$$

and this is true *whatever x is, (even 0, maybe, but who cares?).*

Now, look at:

$$(2 + 3)^2 = 5^2 = 25$$

But $(2 + 3) \times (2 +3) = 2\times2 + 2\times3 + 3\times2 + 3\times3$, by multiplying each pair of numbers.

$$= 2^2 + 2\times2\times3 + 3^2$$

Once again, the numbers don't have to be 2 and 3, they can be anything;

$$(a + b)^2 = a^2 + 2 \times a \times b + b^2$$

In algebra, we leave out the multiplication signs if that wouldn't cause misreading:

$$(a + b)^2 = a^2 + 2ab + b^2$$

The actual letters used don't matter at all:

$$(V + q)^2 = V^2 + 2Vq + q^2$$

There is a set of these *binomial* (two number) equations:

$$(a + b)^2 = a^2 + 2ab + b^2,$$

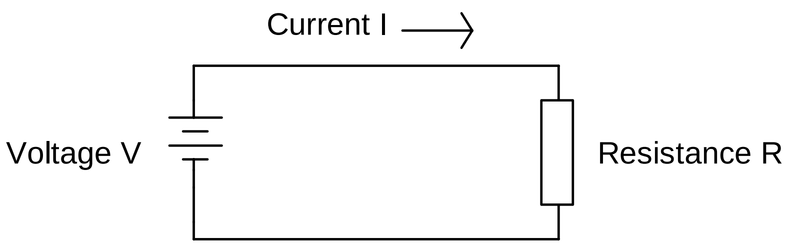
$$(a - b)^2 = a^2 - 2ab + b^2, \text{ and}$$

$$(x + y)(x - y) = x^2 - y^2,$$

as you can prove by multiplying out the left sides as we did above for $(a + b)^2$. For example, if $x = 7$ and $y = 1$, then $(x + y) = 8$ and $(x - y) = 6$. Then $8 \times 6 = 48$, which equals $7^2 - 1^2$. You can sometimes use this to calculate squares with simpler multiplications; $27^2 = (26 \times 28)+1$, and 26×28 is $2 \times 13 \times 2 \times 2 \times 7 = 8 \times 91 = 728$. So $27^2 = 728 + 1 = 729$.

Down to the electronics!

With that background we can begin to tackle Ohm's Law. In words, the voltage V in a simple circuit is equal to the current I multiplied by the resistance R. (Dr. Ohm didn't write it like that, but then that was long ago.)



There are no actual numbers mentioned, so we can use the letters to write the equation that the words mean:

$$V = IR$$

V, I and R are called the 'terms' in the equation. Well, that's jolly good if we know the current and the resistance but we usually know the voltage, and probably the resistance rather than the current– we have to know two things to calculate the third. So, if we know V and R, how can we find I?

Well, we could get rid of the R on the right side by dividing by it, and to keep the equation balanced we have to divide the left side by R as well:

$$V/R = IR/R$$

$$= I$$

Bingo! We can, of course just turn the whole thing round to read $I = V/R$.

If we know the voltage and current , how do we find the resistance? How do we get rid of the I from the right side of $V = IR$? Just divide both sides by it, like we did before:

$$V/I = IR/I$$

$$= R$$

There are only three ways of writing Ohm's Law and we have found all of them:

$$V = IR, \; V/R = I \text{ and } V/I = R.$$

Power

We found the resistance R by dividing the voltage V by the current I. If we multiply them instead, we get the power P:

$$P = VI$$

Just as for the Ohm's Law equations, there are two other versions of this one, which is called Joule's Law:

$$V = P/I \text{ and } I = P/V$$

For example, a 100 W lamp on 230 V mains draws a current of 100/230 = 0.44 A. (Since the lamp is a pure resistance, it doesn't matter that the mains is AC, but if that is still a worry, assume that the lamp is on a 230 V DC supply.)

We can get another very useful set of equations by combining Ohm's Law and Joule's Law. These allow us to have power and resistance in the same equation, which saves a two-step calculation. We form these new equations by using Ohm's Law to write one of the terms in the Joule's Law equation differently. For example:

$$P = VI$$

$$= IR \times I$$

$$= I^2R$$

Clearly, there is a set of these:

$$P = V^2/R,$$

but the others need a bit more care:

$$V^2 = PR,$$

$$\text{so } V = \surd(PR)$$

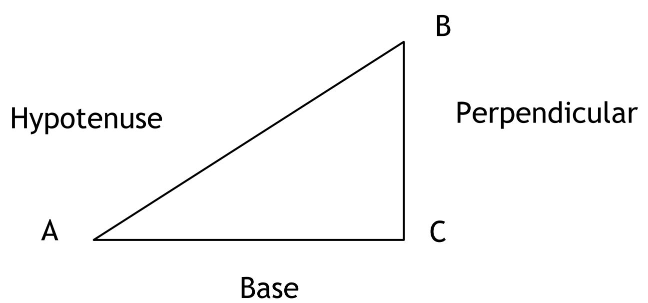
$$I^2 = P/R,$$

$$\text{so } I = \surd(P/R)$$

The brackets in $\surd(PR)$ and $\surd(P/R)$ show that the square root operation applies to both P and R, not just P. (In proper mathematics books, this is shown by a line above the symbols, rather than brackets round them, but Word doesn't do that.)

Angles

Angles come into the picture when we consider AC circuits. The basis of the branch of mathematics called **trigonometry** is simply the angles in a right-angled triangle.



The angle at C is the right-angle (90 degrees), and we are interested in angle A. In particular,

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we are interested in the ratios of the lengths of the sides, because they don't change, however big or small the triangle is, provided the angles don't change.

The ratio Perpendicular/Hypotenuse (P/H) is called the *sine* of angle A, written **sinA**.

The ratio Base/ Hypotenuse(B/H) is called the *cosine* of angle A, written **cosA.**

The ratio Perpendicular/Base (P/B) is called the *tangent* of angle A, written **tanA**.

There are three more ratios, just those turned upside down, which is the same as saying '1 divided by' or 'the reciprocal'. Sine turned upside down becomes cosecant (cosec), 1 divided by a cosine is a secant (sec) and the reciprocal of tangent is cotangent (cot). But you rarely come across these.

Now we saw that we could get extra useful equations by combining Joule's Law and Ohm's Law. Well, by combining the three simple 'trig' equations and adding in Pythagoras (which just says that $P^2 + B^2 = H^2$), we can get literally *thousands* of new equations, a fairly small number of which are actually useful!

What is the connection between the sine and the sine wave? Well, if we take the hypotenuse as a spoke in a wheel whose axle is at A, then the height of the tip of the hypotenuse above the base, which is the perpendicular P, traces out a sine wave as the wheel rotates. No surprise, because H is a fixed length, and P/H is the sine of angle A.

Logarithms and decibels

Remember $3^2 \times 3^3 = 3^5$? What that shows is that using indices we can convert multiplication, which can be difficult, into adding, which is easier. Even better, we can convert division, which is even more difficult, into subtraction. So, if we want to multiply 2 by 7, how can we do it? Well, we use to use books of 'log tables'. These gave the index of 10 that corresponded to any number we looked up (to four or seven places of decimals, usually). For example:

$$2 = 10^{0.3010} \text{ and } 7 = 10^{0.8450}$$

0.3010 is the logarithm of 2 (to base 10) and 0.8450 is the log of 7.

$$\text{So } 2 \times 7 = 10^{0.3010 + 0.8450} = 10^{1.1460}$$

and then we would look up 0.1460 in the 'antilog' section, find the value 1.399, and multiply it by 10 to take account of the leading '1' in '1.1460'. Now, of course we all have calculators. But logarithms are still in wide use, disguised as 'decibels'.

Apart from making big calculations easier (our 2 × 7 example is not very sensible, of course, and the method is much more useful for 2476 × 7794, which would take much longer to multiply out), logarithms simplify the handling of very big and very small numbers, especially those with many digits. These days, we are becoming quite used to nanometres (10^{-9} metres) and terahertz (10^{12} Hz).

For audio signals, we may have less than 1 microvolt or 100 V or more, and the range of currents can be even bigger. Sounds, too, can be very quiet or very loud, and the range of quite common sound pressures covers a ratio of 100 000 to 1. In this case, not only do logarithms help with the arithmetic, but the human ear actually responds logarithmically to sound pressure over a very wide range. To double the subjective loudness of a sound, we have to increase the sound pressure by a ratio of just over 3, not 2.

Originally, the *level in decibels* of a given electrical (telephone) signal was defined in terms of the log of the ratio of two powers:

$$L_{signal} \text{ (in decibels)} = 10 \times \lg(P_{signal}/P_{reference})$$

The factor of 10 comes from the basic unit, the bel, being too big for convenience, so we use decibels - 10 times smaller. 'lg' is the modern symbol for 'log to base 10'

The $P_{reference}$ might be 1 watt or, more likely, 1 mW. **But there must always be a reference value:** a level of, say, '14 dB' with no reference stated or established by a standard or convention, is meaningless.

Soon, the formula was modified to be expressed in terms of voltage or current. Since $P = V^2/R$, or I^2R, the voltages or currents have to be squared. The log of the square of a number is twice the log of the number, so the formula becomes (for voltages):

$$L_{signal} \text{ (in decibels)} = 20 \lg(V_{signal}/V_{reference})$$

Strictly, this should apply ONLY if the R term is the same value for both voltages, but, although many efforts were made to stop decibels being used where the R values are NOT the same, they were unsuccessful. Part of the reason is that different R values rarely cause mistakes in decibel calculations. But it IS necessary to remember that a mistake MIGHT occur. For example, an amplifier that takes an input of 1 V to produce an output of 100 V has a *voltage gain* of 20 lg(100/1) = 40 dB . The load resistance is 100 Ω (making it a 100 W amplifier) and the input resistance is 10 kΩ. So the input *power* with 1 V input is $1^2/10^4$ W = 10^{-4} W and the output power is 10^2 W, giving a ratio of 10^6 and thus a *power gain* of 60 dB.

Exponential or scientific notation

To avoid having to write long numbers like 6285700000000 or 0.000000001457, we split the number into two parts, the basic numerical part and the number of zeroes. The number of zeroes is expressed as a power of 10, so that three zeroes is written as '$\times 10^3$'. The number '11113' can be written '1.1113×10^4'. The numerical part is usually written with one digit to the left of the decimal point, but in engineering, we don't do that; we only use powers of 10 that are multiples of 10^3. This notation is not much of an advantage for '11.13×10^3', but those two biggies come out as '6.2857×10^{12}' and '145.7×10^{-12}'. Note that 10^{-12} means $1/10^{12}$.

Metric prefixes

To avoid similar troubles when dealing with very small or very large numbers of units, we have prefixes that indicate multiples and subdivisions. In engineering, we normally only use multiples and subdivisions in steps of 1000, as mentioned above.

Table of metric multipliers

LARGER than the unit			SMALLER than the unit		
kilo	k	$\times 10^3$	milli	m	$\times 10^{-3}$
mega	M	$\times 10^6$	micro	μ	$\times 10^{-6}$
giga	G	$\times 10^9$	nano	n	$\times 10^{-9}$
tera	T	$\times 10^{12}$	pico	p	$\times 10^{-12}$
peta	P	$\times 10^{15}$	femto	f	$\times 10^{-15}$
exa	E	$\times 10^{18}$	atto	a	$\times 10^{-18}$
zetta	Z	$\times 10^{21}$	zepto		$\times 10^{-21}$
yotta	Y	$\times 10^{24}$	yocto		$\times 10^{-24}$

Full Marx

It has been *seriously* suggested that future prefixes should be suitable versions of groucho/a, chico/a, harpo/a and gummo/a. Zeppo has lost out.

Symbols

Symbols for voltage, current etc., and for units of measurement are part of our technical language and should be used correctly, otherwise it may come out as teknikal langwidj.

Table of symbols

UNIT	SYMBOL	UNIT	SYMBOL	UNIT	SYMBOL
ampere	A	kilohertz	kHz	nanofarad	nF
centimetre	cm	kilohm	kΩ	nanosecond	ns
decibel	dB	litre	L	ohm	Ω
degree (plane angle)	°	megahertz	MHz	pascal	Pa
farad	F	metre	m	picofarad	pF
gauss	Gs	microfarad	μF	second (time)	s (small!)
gram	g	micrometre	μm	siemens	S (Capital!)
henry	H	microsecond	μs	tesla	T
hertz	Hz	milliampere	mA	volt	V
hour	h	millihenry	mH	watt	W
inch	in	millimetre	mm	weber	Wb
joule	J	millivolt	mV		
kelvin	K	minute (time)	min		

Symbols that are based on a person's name have a capital letter, but the *unit* name doesn't. So 5 W = 5 watts. There is also a convention that the unit names don't have 's' in the plural, but it isn't generally followed. However, DON'T write 'henries', its 'henrys' or just 'henry'.

Further Reading List

Revise GCSE: Maths Letts, ISBN 1858059321 £9.99

Revise AS & A2: Maths Letts, Available from: 04/01/2005 ISBN 1843154773 £14.99

The Art of Electronics (2nd Ed.), P. Horowitz and Hill W., Cambridge University Press,

ISBN 0 521 37095 7