AC circuits, sine waves, impedance and admittance

The Institute of Sound and Communications Engineers

J. M. Woodgate F Inst SCE

1 AC and d.c.

DC is (or appears to be) easy: it flows constantly until you switch off. Voltage and current are simple, fixed values and there is only one sort of each. AC shuffles backwards and forwards: voltage and current vary with time and can be expressed in several different ways. We also have frequency, phase angle and waveform to take into account. But we wouldn't be here if we did not have a.c., so we *have* to understand it.

Frequency is the number of times the current changes direction in one second. It is measured in hertz (Hz).

Phase angle is a measure of how far out of step two voltages or currents *of the same frequency* are. We cannot assign a phase angle to two signals of different frequencies.

Waveform is the shape of the graph of a voltage or current against time. It is what an ordinary oscilloscope displays.

2 Sine waves

What's so special about sine waves? Mathematicians have proved that you can build up any given waveform out of waves of almost any shape, so it's not that. What *is* unique is that if you apply a sinusoidal voltage to *any* combination of resistors, inductors and capacitors, *no new waveforms appear anywhere*. It is this property that leads us to say that a sine-wave signal 'contains only one frequency'.

Now we need to look at a sinusoidal voltage (or current) waveform. Because the waveform is symmetrical about the zero line, the long-term average voltage is zero, which is not a useful conclusion. The average value over a half-cycle can be shown to be $2/\pi$ times the peak value. This is important for rectifier circuits. But for most purposes, what we need to know is the measure of voltage that relates to the power in the circuit. Let us take a circuit that has only resistance R. Then at any instant, the power is given by Joule's Law:

$$p = u^2/R$$

We use lower case italic letters for instantaneous values. So, to get the total power P in the circuit, we would add up all these instantaneous power values over one cycle of the sine wave and find the value of voltage to put into the Joule's Law equation to obtain that total power. Luckily, the mathematical process of 'integration' allows us to do that, and the result is:

$$P = (\hat{U}/\surd 2)^2\ /R$$

where $\hat{U}$ is the peak voltage. This value $\hat{U}/\surd 2$ is called the 'root-mean-square' (r.m.s.) voltage, and is normally written just 'U'. Equally, 'I' means the r.m.s. current, unless the text specifically says otherwise.

Being based on power and Joule's Law, an alternating current of a given r.m.s. value produces as much heat when *flowing through* a resistor as a direct current of the same value. We can equally say that an alternating voltage of a given r.m.s. value produces as much heat when

DISCLAIMER (we have to do this)

Care is taken to determine that 'Training Notes' do not refer to any copyrighted or patented circuit or technique, but ISCE can accept no responsibility in this connection. Users of the information in a 'Training Note' must satisfy themselves that they do not infringe any Intellectual Property Rights.

developed across a resistor as a direct voltage of the same value. It is important not to think of voltage as flowing, or current as static.

3 Impedance - Episode 1

When we apply a sinusoidal voltage to an inductor or capacitor, a current flows. The current is determined by the *impedance Z* of the inductor or capacitor. To study this, we need the mathematical expression for a sinusoidal voltage as a function of time:

$$u = \hat{U} \sin 2\pi f t$$

where f is the frequency in hertz (number of cycles per second) and t is the time in seconds. We usually use the symbol ω (lower case omega) for $2\pi f$. ω is called the 'angular frequency'.

To deal with a.c. circuits, we extend Ohm's Law to:

$$U = IZ$$

The impedance Z of an ideal (lossless) inductor, which can also be called 'inductive reactance' X_L', is proportional to its *inductance L* and to frequency:

$$X_L = \omega L$$

Inductance is measured in henrys, symbol H.

For an air-cored coil, where the loss is only due to resistance R of the winding wire, the impedance Z is given by:

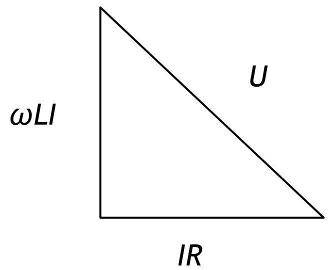
$$Z = \surd\{R^2 + (\omega L)^2\}$$

Why? When we apply a sinusoidal voltage to a resistor, the current flows exactly in step with the voltage. But when we substitute an inductor or capacitor, that is no longer true.

4 Phase angle

When we apply a sinusoidal voltage to an inductor, it takes time and energy to build up the magnetic field. This results in the current waveform being delayed in time with respect to the voltage waveform. For an ideal inductor, the delay or 'lag' is a quarter of a cycle, or, since a whole cycle can be expressed as 360°, it is 90°. If we have an air-cored inductor, it can be modelled as an ideal inductor in series with its resistance. Now we have the same current through both, but the voltage across the resistor is in step, or 'in phase' with the current while that across the ideal inductor *leads* the current by 90°. We can illustrate that with what used to be called a vector diagram but now has to be called a phasor diagram. Let us hope that the phasors are not set to 'STUN'. The phasor diagram for the voltages is a right-angled triangle, and the resulting voltage U is given by Pythagoras:

$$U = \surd\{(IR)^2 + (\omega LI)^2\}$$



5 Impedance - Episode 2

Applying the extended 'Ohm's Law for a.c.', we get:

DISCLAIMER (we have to do this)

Care is taken to determine that 'Training Notes' do not refer to any copyrighted or patented circuit or technique, but ISCE can accept no responsibility in this connection. Users of the information in a 'Training Note' must satisfy themselves that they do not infringe any Intellectual Property Rights.

$$Z = U/I$$
$$= \sqrt{R^2 + (\omega L)^2}$$

which is what was sprung on you above as a nice surprise.

When we think about applying a sinusoidal voltage to an ideal capacitor, there may be an intuitive problem that is there also but doesn't seem to bug people when considering applying a voltage to an inductor. (It may well bug them in practice when they try to switch off, though!) The ideal capacitor is an empty storage jar and has no resistance, So even the tiniest voltage would appear to be able to drive an infinite current through it.

The problem comes from the fact that a mathematical sine wave started with the Big Bang and goes on for ever: it was never switched on and will never be switched off. So if we think about *applying* a voltage to something, what we are applying is not truly a sine wave. What happens is that for a few cycles, things ('transients') have to settle down, and after that everything is well-behaved. What we then find is that the current *leads* the voltage by 90°, just the opposite to the situation with the inductor. So the phasor diagram for a lossy capacitor, modelled as an ideal one in series with its ESR (equivalent series resistance) is a right-angled triangle just as for the inductor, but flipped over about the '*IR*' side, and the impedance equation looks very similar.

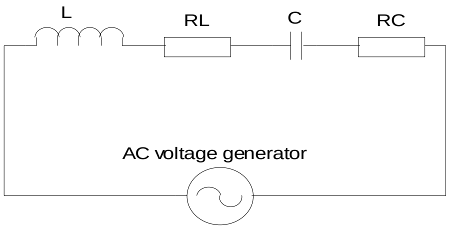
It isn't identical, though, because the impedance or 'capacitive reactance' X_C' of an ideal capacitor is *inversely* proportional to both the capacitance value in farads, and the frequency:

$$X_C = 1/\omega C$$

So we get for the impedance of a lossy capacitor, whose losses can be fairly represented as series resistance:

$$Z = \sqrt{R^2 + (1/\omega C)^2}$$

Now, suppose we connect an air-cored inductor in series with the capacitor.



We have to account for the fact that the voltages across the ideal components are 180° out of phase by including a minus sign between them:

$$Z = \sqrt{(R_L + R_C)^2 + (\omega L - 1/\omega C)^2}$$

You can see that an interesting thing happens when $\omega L = 1/\omega C$. The reactances add up to zero and the current is controlled *only* by the resistances. This condition is called *series resonance*. Because the current at this *resonance frequency* is high, the voltages across the inductor and capacitor are also high, maybe much higher than the applied voltage. We can get a similar effect when an inductor and capacitor are connected in parallel. This *parallel resonance* results in the impedance of the circuit being much *higher* at the resonance frequency than at other frequencies. This can be seen if we consider that *at a fixed frequency,* we can replace an inductor or capacitor in series with a low-value resistor into an inductor or capacitor in parallel with a high-value resistor. So, in the parallel *tuned circuit* at

DISCLAIMER (we have to do this)

Care is taken to determine that 'Training Notes' do not refer to any copyrighted or patented circuit or technique, but ISCE can accept no responsibility in this connection. Users of the information in a 'Training Note' must satisfy themselves that they do not infringe any Intellectual Property Rights.

resonance, we have just the two high-value *parallel equivalent resistances,* in parallel.

In general, we can model a lossy inductor or capacitor over a wide range of frequencies by an ideal component in series with a low-value resistor, and a high-value resistor in parallel with the series combination

6 Admittance

Especially for calculations on parallel circuits and at high radio frequencies, *admittance Y,* the reciprocal of impedance *Z*, is useful. The corresponding *inductive* and *capacitive susceptances* $B_L = 1/\omega L$ and $B_C = \omega C$. The reciprocal of resistance is *conductance G,* and these reciprocal quantities are measured in siemens (NOT 'siemen'), symbol S. (So DON'T use 'S' for 'seconds', the correct symbol is 's'.)

DISCLAIMER (we have to do this)

Care is taken to determine that 'Training Notes' do not refer to any copyrighted or patented circuit or technique, but ISCE can accept no responsibility in this connection. Users of the information in a 'Training Note' must satisfy themselves that they do not infringe any Intellectual Property Rights.